Existence and Computation of Infinite Horizon Model Predictive Control with Active Steady-State Constraints

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Abstract

This paper addresses the existence and implementation of the infinite horizon controller for the case of active steady-state constraints. The case of active steady-state constraints is important because, in many practical applications, controllers are required to operate at the boundary of the feasible region (for instance, in order to maximize global economic objectives). For this case, the usual finite horizon parameterizations with terminal cost cannot be applied since the origin lies on the boundary of the feasible region, and only suboptimal solutions are available.

We propose here an iterative algorithm that generates an upper bound and a lower bound finite horizon approximation to the optimal solution. We show convergence of these boundary approximations to the optimal solution as the horizon increases is shown. The difference between the upper and lower bound solutions is used to bound the difference between the approximating solution and the optimal one. The algorithm provides a solution that is guaranteed to be within a user specified tolerance of the optimal solution. A numerical example with comparison between optimal and suboptimal controllers is presented.

Index terms – Model Predictive Control, Steady-state constraints, Optimal Control

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1 Introduction

Model Predictive Control (MPC) is a technique in which a process model is used to forecast future process behavior, and the sequence of future control inputs is computed as the solution to an open-loop optimization problem. The first element of the optimal input sequence is used as the process input. The remaining elements of the input sequence are discarded and the optimization is repeated at each sampling time. Feedback from measurements is considered by correcting the model prediction, based on the error between the measurement and prediction. Many methods are available for this correction. Several recent reviews [1, 2, 3] summarize the theoretical formulations and industrial implementations of MPC.

In this paper, we consider the infinite horizon formulation of model predictive control (IHMPC) and address the case of constraints that are active at steady state. Process constraints arise both from physical limitations (for example, a valve can be at maximum fully open and at minimum totally closed) and from safety and performance specifications. Most papers on constrained infinite horizon MPC rely on the assumption that the origin is in the interior of the feasible region [4, 5, 6, 7, 8]. It is frequently the case, however, that in order to maximize performance objectives, the MPC controller operates at the boundary of the feasible region with respect to both input and output constraints. Moreover, when a nonzero disturbance enters the process, it is often the case that one or more manipulated inputs ride at their corresponding saturation values during a period of steady-state operation. These cases give rise to problem formulations in which the origin lies on the boundary of the feasible region. This situation was treated in [9], which provides a suboptimal solution for this problem. The main contribution of this paper is to provide an algorithm for finding the optimal solution of the constrained infinite horizon optimization problem.

The paper is organized as follows. In Section 2, we recall the formulation of the infinite horizon controller, presenting the common finite parameterization with terminal constraint and discussing feasibility limitations. In Section 3, we discuss the proposed algorithm and prove its convergence to the optimal solution. Some implementation issues are addressed in Section 4 and we present an application of this method in Section 5. Finally, in Section 6, we summarize the main results of this work. Some definitions and results for infinite dimensional quadratic programs are reported in Appendix A, while some existence issues are addressed in Appendix B.

2 Formulation of the problem

2.1 Infinite Horizon Model Predictive Control

In this paper we consider time-invariant, linear, discrete systems described by the following model:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \tag{2.1}$$

in which $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the input vector and $y_k \in \mathbb{R}^p$ is the output vector. It is assumed that the pair (A, B) is stabilizable and the pair (C, A) is detectable.

When a complete measurement of the state vector is not available, a state estimator is required. Estimation is not germane to the subject of this paper and we assume simply that a Kalman filter is used, which provides at each sampling time an estimate of x_k given the plant measurements y_k. In the presence of plant-model mismatch or unmodeled disturbances, a disturbance model is required to obtain offset-free control. Several kinds of disturbance models can be used to achieve this goal. Here we consider the well known output disturbance model defined by the following

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ p_{k+1} &= p_k, \\ y_k &= Cx_k + p_k. \end{aligned} \tag{2.2}$$

Given the current disturbance estimate, p_k, the steady-state targets are computed by solving the following optimization problem [9]:

$$\min_{x_{s,k}, u_{s,k}, y_{s,k}, \eta} \frac{1}{2} \left\{ \eta^T Q_s \eta + (u_{s,k} - \bar{u})^T R_s (u_{s,k} - \bar{u}) \right\} + q_s^T \eta \tag{2.3}$$

subject to:

$$\begin{gathered} x_{s,k} = Ax_{s,k} + Bu_{s,k}, \\ y_{s,k} = Cx_{s,k} + p_k, \\ \bar{y} - \eta \le y_{s,k} \le \bar{y} + \eta, \qquad \eta \ge 0, \\ u_{\min} \le u_{s,k} \le u_{\max}, \qquad y_{\min} \le y_{s,k} \le y_{\max}, \end{gathered}$$

in which Q_s and R_s are positive definite matrices, $\bar{y}$ and $\bar{u}$ are the desired targets for output and input, respectively. These targets may be computed by a global economic optimizer, which supervises the MPC controller. We assume that $y_{\min} < y_{\max}$, $u_{\min} < u_{\max}$. For appropriate choice of q_s, the linear penalty $q_s^T \eta$ guarantees that the output slack variable η is zero whenever it is possible to use this value without violating feasibility; that is, whenever the target value $\bar{y}$ is a feasible choice for $y_{s,k}$.

The control action is computed from the minimization of the following infinite horizon quadratic objective function:

$$\begin{aligned} \mathcal{O}(N): \quad \min_{\{x_{k+j}, u_{k+j}, y_{k+j}, \Delta u_{k+j}\}_{j=0}^{\infty}} \frac{1}{2} \sum_{j=0}^{\infty} &(y_{k+j} - y_{s,k})^T Q (y_{k+j} - y_{s,k}) + \\ &(u_{k+j} - u_{s,k})^T R (u_{k+j} - u_{s,k}) + \Delta u_{k+j}^T S \Delta u_{k+j} \end{aligned} \tag{2.4a}$$

subject to

$$x_{k+j+1} = Ax_{k+j} + Bu_{k+j}, \quad j = 0, 1, 2, \ldots, \tag{2.4b}$$

$$y_{k+j} = Cx_{k+j} + p_k, \quad j = 0, 1, 2, \ldots, \tag{2.4c}$$

$$u_{\min} \le u_{k+j} \le u_{\max}, \qquad j = 0, 1, 2 \ldots, \tag{2.4d}$$

$$-\Delta u_{\max} \le \Delta u_{k+j} \le \Delta u_{\max}, \qquad j = 1, 2, \ldots, \tag{2.4e}$$

$$y_{\min} \le y_{k+j} \le y_{\max}, \qquad j = 0, 1, 2, \ldots, \tag{2.4f}$$

in which $\Delta u_{k+j} \triangleq u_{k+j} - u_{k+j-1}$ and $\Delta u_{\max} > 0$. We assume that Q is positive definite, R and S are positive semidefinite, and R or S is positive definite.

By using the steady-state targets, we have that

$$y_{k+j} - y_{s,k} = C(x_{k+j} - x_{s,k}) \tag{2.5}$$

Thus, we rewrite the infinite horizon objective function in a more convenient form. Consider the following new variables:

$$\begin{gathered} w_j = \begin{bmatrix} x_{k+j} - x_{s,k} \\ u_{k+j-1} - u_{s,k} \end{bmatrix}, \qquad v_j = u_{k+j} - u_{s,k}, \qquad A \leftarrow \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad B \leftarrow \begin{bmatrix} B \\ I \end{bmatrix}, \\ C \leftarrow \begin{bmatrix} C & 0 \end{bmatrix}, \qquad Q \leftarrow \begin{bmatrix} C^T Q C & 0 \\ 0 & S \end{bmatrix}, \qquad R \leftarrow R + S, \qquad M = \begin{bmatrix} 0 \\ -S \end{bmatrix}, \\ D = \begin{bmatrix} I \\ -I \end{bmatrix}, \quad d = \begin{bmatrix} u_{\max} - u_{s,k} \\ -u_{\min} + u_{s,k} \end{bmatrix}, \quad E = \begin{bmatrix} I \\ -I \end{bmatrix}, \quad G = \begin{bmatrix} 0 & I \\ 0 & -I \end{bmatrix}, \quad e = \begin{bmatrix} \Delta u_{\max} \\ \Delta u_{\max} \end{bmatrix}, \\ H = \begin{bmatrix} C & 0 \\ -C & 0 \end{bmatrix}, \quad h = \begin{bmatrix} y_{\max} - y_{s,k} \\ -y_{\min} + y_{s,k} \end{bmatrix}. \end{gathered} \tag{2.6}$$

It follows from (2.6) that the "new" R is positive definite since either the "old" R or the "old" S is positive definite. From the steady-state target calculation (2.3) we have that $d \ge 0$, $h \ge 0$. Moreover, since $\Delta u_{\max} > 0$, we have that $e > 0$. By substituting into (2.4), we find that the optimal control problem requires the minimization of the following infinite horizon objective function:

$$\mathcal{O}(N): \qquad \min_{\{w_j, v_j\}_{j=0}^{\infty}} \frac{1}{2} \sum_{j=0}^{\infty} w_j^T Q w_j + v_j^T R v_j + 2 w_j^T M v_j \tag{2.7a}$$

subject to:

$$w_0 = w^{\text{init}}, \qquad w_{j+1} = Aw_j + Bv_j, \qquad j = 0, 1, 2, \ldots, \tag{2.7b}$$

$$Dv_j \le d, \qquad j = 0, 1, 2, \ldots, \tag{2.7c}$$

$$Ev_j - Gw_j \le e, \qquad j = 0, 1, 2, \ldots, \tag{2.7d}$$

$$Hw_j \le h, \qquad j = 0, 1, 2, \ldots, \tag{2.7e}$$

The optimization problem (2.7) may be infeasible due to the presence of input and state constraints. For example, a disturbance may enter the plant and cause the current state w_0 to leave the feasible region. The problem of feasibility for MPC has been addressed in several ways [6, 10, 11, 12]. Any of these state constraint softening approaches can be incorporated into the methodology proposed here.

Therefore, we assume that, given a w^{init}, a sequence $\{v_j, w_j\}_{j=0}^{\infty}$ exists, which is feasible with respect to constraints (2.7b) (2.7c), (2.7d), (2.7e) and gives a finite value of the objective function in (2.7). From a physical point of view, we assume that a sequence of inputs exists that is able to bring the system to the origin while respecting input and state constraints. This assumption is commonly referred to as constrained stabilizability. We also assume that the pair $(Q^{1/2}, A)$ is detectable. This assumption prevents unstable modes evolving without appearing in the objective function. This assumption implies that, if a feasible sequence exists, then $w_j \to 0$, $v_j \to 0$ as $j \to \infty$ [8].

2.2 Finite parameterization of the optimal control problem

The infinite horizon problem in (2.7) was addressed in several ways [5, 4, 13, 7, 8]. The key step in these analyses is to recognize that inequality constraints remain active only for a finite number of sampling times while the states and the inputs are approaching the origin.

The solution of the unconstrained infinite horizon problem is the well known linear feedback control law:

$$v_j = -Kw_j, \tag{2.8}$$

in which K is computed from the solution of the discrete-algebraic Riccati equation. For nonlinear systems, Michalska and Mayne [14] present the "dual-mode" controller in which the optimal linear control law in (2.8) is appended to the input sequence after a finite horizon. The same idea is used for linear systems [7, 8], where the following finite horizon objective function is used as replacement of (2.7):

$$\min_{\{w_j\}_{j=0}^{N}, \{v_j\}_{j=0}^{N-1}} \frac{1}{2} \sum_{j=0}^{N-1} \left\{ w_j^T Q w_j + v_j^T R v_j + 2 w_j^T M v_j \right\} + \frac{1}{2} w_N^T \Pi w_N \tag{2.9a}$$

subject to:

$$w_0 = w^{\text{init}}, \qquad w_{j+1} = Aw_j + Bv_j, \qquad j = 0, 1, 2, \ldots, N-1, \tag{2.9b}$$

$$Dv_j \leq d, \qquad j = 0, 1, 2, \ldots N-1, \tag{2.9c}$$

$$Ev_j - Gw_j \leq e, \qquad j = 0, 1, 2, \ldots N-1, \tag{2.9d}$$

$$Hw_j \leq h, \qquad j = 0, 1, 2, \ldots N, \tag{2.9e}$$

in which the cost-to-go Π is the solution of the discrete-algebraic Riccati equation. In addition to constraints (2.9c), (2.9d) and (2.9e), the final state w_N is required to be in the following positive invariant convex set [15]:

$$\mathbb{O} = \left\{ w | \tilde{H}(A - BK)^i w \leq \tilde{h} \qquad \forall i \geq 0 \right\}, \tag{2.10}$$

in which

$$\tilde{H} = \begin{bmatrix} -DK \\ -(EK+G) \\ H \end{bmatrix}; \qquad \tilde{h} = \begin{bmatrix} d \\ e \\ h \end{bmatrix}.$$

If the final state w_N is in the invariant set defined by (2.10), the optimal unconstrained control law (2.8) yields a solution that satisfies state and input constraints at all future sampling times. When the origin is in the relative interior of the feasible region, which is true if and only if d, e and h are strictly positive, the existence of a nontrivial invariant set is guaranteed.

When the origin lies inside the feasible region for (2.7), we may construct a solution for the infinite-horizon problem (2.7) from the solution of the finite-horizon problem (2.9) provided that the horizon index N is sufficiently large. Typically, one solves (2.9) for some N and then checks to see whether w_N lies in the output admissible set $\mathbb{O}$. If so, it can be shown that the optimal values of w_j, $j = 0, 1, 2, \dots, N$ and v_j, $j = 0, 1, 2, \dots, N-1$ are identical for (2.7) and (2.9). Otherwise, one increases the value of N in (2.9) and repeats the process.

When state constraints are active at steady state, arbitrarily small constant disturbances would render the hard constrained problem infeasible, which means that there is no feasible sequence that brings the system to the origin without persistently violating the active constraints permanently. Therefore, we assume that state constraints are not active at steady state. In other words, it is assumed that the elements of h (but not of d) in (2.7e) or (2.9e) are all strictly positive. Moreover, velocity constraints are never active at steady state, since e is positive by definition.

3 Optimal solution of the infinite horizon problem

In this section, we introduce a method for finding an approximate solution of the problem (2.7) in the case in which some constraints are active at steady state, so that the origin lies on the boundary of the feasible region. Our approach is to construct two problems that approximate (2.7) and for which solutions can be calculated – one of which has an optimal objective value that is an upper bound for the optimal objective of (2.7) and the other a lower bound. By showing that these two bounds approach each other as $N \to \infty$, we obtain increasingly accurate estimates of the optimal objective for (2.7). Moreover, we use the difference between the upper and lower bound objective functions to obtain a bound on the difference between the solution of the approximating problem and the solution of (2.7).

3.1 Upper bound on the optimal solution

An upper bound on the optimal objective Φ^* of (2.7) can be computed by using the method proposed in [9]. In this approach, a suboptimal solution to (2.7) is found by restricting the evolution of the input and state trajectories to the null space of the active steady-state constraints, after the finite horizon $N > 0$. This solution is found by minimizing the following infinite horizon objective function:

$$\mathcal{U}(N): \quad \min_{\{w_j,v_j\}_{j=0}^{\infty}} \frac{1}{2}\sum_{j=0}^{\infty} w_j^T Q w_j + v_j^T R v_j + 2 w_j^T M v_j \tag{3.1a}$$

$$\text{subject to } (2.7\text{b}), (2.7\text{c}), (2.7\text{d}), (2.7\text{e}) \tag{3.1b}$$

and

$$\bar{D} v_j = 0, \qquad j = N, N+1, \ldots, \tag{3.1c}$$

where $\bar{D}$ denotes the row sub-matrix of D corresponding to input inequality constraints active at steady state, that is the rows of D whose corresponding elements of d are zero.

Let Φ_N^u be the optimal objective value for $\mathcal{U}(N)$ in (3.1). Since (3.1) has more constraints than (2.7), its feasible region is smaller, so we have:

$$\Phi^* \leq \Phi_N^u. \tag{3.2}$$

We can reformulate the infinite horizon problem (3.1) as a finite-horizon problem that can be solved by practical means as follows:

$$\min_{\{w_j\}_{j=0}^{N}, \{v_j\}_{j=0}^{N-1}} \frac{1}{2}\sum_{j=0}^{N-1} \left\{ w_j^T Q w_j + v_j^T R v_j + 2 w_j^T M v_j \right\} + \frac{1}{2} w_N^T \bar{\Pi} w_N \tag{3.3a}$$

$$\text{subject to } (2.9\text{b}), (2.9\text{c}), (2.9\text{d}), (2.9\text{e}) \tag{3.3b}$$

where the cost-to-go matrix $\bar{\Pi}$ is associated with the unconstrained control law:

$$v_j = -\bar{K} w_j. \tag{3.4}$$

The computation of $\bar{K}$ is described in [9]. In order for such a linear control law to exist, the system must be stabilizable in the null space of the active steady-state constraints. In some cases, this condition requires the controller to zero the unstable modes that are not controllable in this subspace. When all input constraints are active at steady state, it is necessary to zero all the unstable modes and the upper bounding problem results in the controller discussed in [13]. In Appendix B it is shown that the upper bounding problem (3.3) is feasible for N sufficiently large under the assumption of the existence of a feasible sequence for the optimal problem (2.7).

The problems (3.1) and (3.3) are identical in the sense that the solution components $v_0, v_1, \ldots, v_{N-1}$ are the same for each. We compute the remaining components by using the unconstrained evolution of the system under the feedback gain $\bar{K}$:

$$v_j = -\bar{K} w_j \qquad j = N, N+1, \ldots, \tag{3.5a}$$

$$w_{j+1} = A w_j + B v_j \qquad j = N, N+1, \ldots \tag{3.5b}$$

If the final state w_N does not lie in the output admissible set for the subset of inequalities not active at steady state under the feedback gain $\bar{K}$ ([9], [15]), the horizon N must be increased in order for the solution components of (3.1) and (3.3) to be equal.

3.2 Lower bound on the optimal solution

A lower bound on the optimal objective Φ^* of (2.7) can be found by minimizing the following infinite horizon objective function:

$$\mathcal{L}(N): \qquad \min_{\{w_j,v_j\}_{j=0}^{\infty}} \frac{1}{2}\sum_{j=0}^{\infty} w_j^T Q w_j + v_j^T R v_j + 2 w_j^T M v_j \tag{3.6a}$$

$$\text{subject to (2.7b), (2.9c), (2.9d), (2.9e).} \tag{3.6b}$$

Notice that constraints (2.9c), (2.9d), (2.9e) are enforced over a finite horizon N only.

Let Φ_N^l be the optimal objective value for $\mathcal{L}(N)$. Since (3.6) has fewer constraints than (2.7), it is clear that

$$\Phi_N^l \le \Phi^*, \qquad \forall N > 0. \tag{3.7}$$

The infinite horizon problem (3.6) can be solved by using the finite parameterization in (2.9), without adding the constraint on the final state. That is, we solve the following problem:

$$\min_{\{w_j\}_{j=0}^{N},\{v_j\}_{j=0}^{N-1}} \frac{1}{2}\sum_{j=0}^{N-1}\left\{w_j^T Q w_j + v_j^T R v_j + 2 w_j^T M v_j\right\} + \frac{1}{2} w_N^T \Pi w_N \tag{3.8a}$$

$$\text{subject to (2.9b), (2.9c), (2.9d), (2.9e).} \tag{3.8b}$$

The solution components $v_0, v_1, \dots, v_{N-1}$ are the same for (3.6) and (3.8). We obtain $v_N, v_{N+1}, \dots,$ from (3.8) by using the unconstrained evolution:

$$v_j = -K w_j \qquad j = N, N+1, \dots, \tag{3.9a}$$

$$w_{j+1} = A w_j + B v_j \qquad j = N, N+1, \dots \tag{3.9b}$$

3.3 Convergence of the Optimal Sequences

In this section we show that v_0 obtained from each of the three problems (2.7), (3.1), (3.6) converges to the same point as $N \to \infty$. We prove the result by treating (2.7), (3.1), (3.6) as strictly convex quadratic programs in the variable

$$z = (v_0, v_1, \dots) \in \ell^2,$$

where

$$\begin{aligned} \ell^2 &= \left\{ z = (z_1, z_2, \dots) \,|\, z_i \in \mathbb{R},\ i = 1, 2, \dots;\ \sum_{i=1}^{\infty} z_i^2 < \infty \right\} \\ &= \left\{ (v_0, v_1, v_2, \dots) \,|\, v_j \in \mathbb{R}^m,\ j = 0, 1, \dots;\ \sum_{j=0}^{\infty} \|v_j\|_2^2 < \infty \right\}. \end{aligned} \tag{3.10}$$

By using the state equation to eliminate w_j for $j = 1, 2, \ldots$, all three problems (2.7), (3.1), (3.6) have the following form:

$$\min_z f(z) = \frac{1}{2}\langle z, Uz\rangle + \langle c, z\rangle, \qquad \text{subject to } z \in \mathcal{C}, \tag{3.11}$$

where $\mathcal{C}$ is a closed, convex subset of ℓ^2. See Appendix A.1 for further details and results on quadratic programs on the space ℓ^2.

Note that the restriction (3.10) does not hamper our ability to consider interesting points. An input sequence $\{v_j = u_{k+j} - u_{s,k}\}_{j=0}^{\infty}$ for which $\sum_{j=0}^{\infty} \|v_j\|_2^2 = \infty$ is such that $\lim_{j\to\infty} u_{k+j} \neq u_{s,k}$, which implies that $\lim_{j\to\infty} y_{k+j} \neq y_{s,k}$ (this property follows from unicity of the solution of the target calculation problem [9]). Therefore, the objective function in (2.4) (and the equivalent one in (2.7)) would be infinite, since Q in (2.4) is positive definite and all the other terms are nonnegative.

In describing the limiting behavior of the solutions of the problems (3.1) and (3.6) as $N \to \infty$, we use results from Appendix A.2 concerning the minimizers of a strictly convex quadratic function over increasing and decreasing sequences of sets. We consider first the lower-bounding problem (3.6). Let $\hat{\mathcal{C}}_N$ denote the feasible set in ℓ^2 for this problem; that is, the set of vectors $v = (v_0, v_1, v_2, \ldots)$ for which there is a $w = (w_0, w_1, w_2, \ldots)$ such that (v, w) satisfy (3.6b) and, in addition, $\sum_{j=0}^{\infty} \|v_j\|_2^2 < \infty$. We obtain (3.6) by setting $\mathcal{C} = \hat{\mathcal{C}}_N$ in (3.11). It is clear that $\{\hat{\mathcal{C}}_N\}_{N=1,2,3\ldots}$ is a decreasing sequence of sets. Moreover, it is easy to see that the set defined by

$$\hat{\mathcal{C}} = \bigcap_{N=1,2,\ldots} \hat{\mathcal{C}}_N$$

is simply the feasible set for (2.7); that is, the set of vectors $v = (v_0, v_1, v_2, \ldots)$ for which there is an $w = (w_0, w_1, w_2, \ldots)$ such that (v, w) satisfy (2.7b), (2.7c), (2.7d), (2.7e) and, in addition, $\sum_{j=0}^{\infty} \|v_j\|_2^2 < \infty$. We obtain (2.7) by setting $\mathcal{C} = \hat{\mathcal{C}}$ in (3.11).

We introduce the notation $\hat{z}_N = (\hat{v}_{N,0}, \hat{v}_{N,1}, \hat{v}_{N,2}, \ldots)$ for the minimizer of (3.11) with $\mathcal{C} = \hat{\mathcal{C}}_N$ (equivalently, (3.6)), and $z^* = (v_0^*, v_1^*, v_2^*, \ldots)$ for the minimizer of (3.11) with $\mathcal{C} = \hat{\mathcal{C}}$ (equivalently, (2.7)). By applying Theorem A.4 from Appendix A.2, we have that

$$\lim_{N\to\infty} \hat{z}_N = z^*, \qquad \Phi_N^l \uparrow \Phi^*, \tag{3.12}$$

where the last limit indicates that the sequence of optimal objective values $\{\Phi_N^l\}_{N=1,2,\ldots}$ for (3.6) is increasing and approaches the optimal objective Φ^* for (2.7) as $N \to \infty$.

We turn now to the upper-bounding problem (3.1). Let $\bar{\mathcal{C}}_N$ denote the feasible set in ℓ^2 for this problem; that is, the set of vectors $v = (v_0, v_1, v_2, \ldots)$ for which there is an $w = (w_0, w_1, w_2, \ldots)$ such that (v, w) satisfy (3.1b), (3.1c), and, in addition, $\sum_{j=0}^{\infty} \|v_j\|_2^2 < \infty$. We obtain (3.1) by setting $\mathcal{C} = \bar{\mathcal{C}}_N$ in (3.11). It is clear that $\{\bar{\mathcal{C}}_N\}_{N=1,2,3\ldots}$ is an increasing sequence of sets. Its limit, defined by

$$\bar{\mathcal{C}} = \bigcup_{N=1,2,\ldots} \bar{\mathcal{C}}_N$$

is the set of vectors $v \in \ell^2$ for which there is an w such that (v, w) satisfy (3.1b), as well as satisfying (3.1c) *for some value of* N. This is not the same set as the feasible set $\hat{\mathcal{C}}$ for (2.7), in which the restriction (3.1c) does not appear at all. Hence, the issues in considering the upper-bounding problem are slightly more subtle than for the lower-bounding problem, and are considered in our discussion of increasing sequences of sets in Appendix A.2. The remainder of our discussion below shows that we can apply Theorem A.5 in Appendix A.2 to this case, and arrive at the desired conclusion that the sequence of minimizers of the upper-bounding problem (3.1) converges to the minimizer of the optimal problem (2.7).

We can identify the feasible set $\hat{\mathcal{C}}$ for (2.7) with the set $\mathcal{C}^*$ in (A.20), and identify the minimizer z^* of (2.7) with $\bar{z}$ of (A.20). Note that $\hat{\mathcal{C}}$ is certainly closed, and that since $\bar{\mathcal{C}}_N \subset \hat{\mathcal{C}}$ for every N we certainly have $\bar{\mathcal{C}} \subset \hat{\mathcal{C}}$. It remains only to show that $z^* \in \mathrm{cl}(\bar{\mathcal{C}})$; that is, the solution of (2.7) lies in the closure of the set formed by the union of the feasible sets for (3.1), over all N.

To show that z^* lies in the closure of $\bar{\mathcal{C}}$, we construct a sequence $\{z_N\}$ such that

$$z_N \in \bar{\mathcal{C}}_N \subset \bar{\mathcal{C}}, \quad \text{for all } N \text{ sufficiently large, and } z_N \to z^*.$$

Writing

$$z^* = (v_0^*, v_1^*, v_2^*, \dots),$$

we have by the definition of $\hat{\mathcal{C}}$ that there is a vector $w^* = (w_0^*, w_1^*, w_2^*, \dots)$ such that (v^*, w^*) satisfies the conditions (2.7b), (2.7c), (2.7d), (2.7e). Since $z^* \in \ell^2$, we also have that

$$\lim_{N \to \infty} \sum_{j=N}^{\infty} \|v_i^*\|_2^2 = 0. \tag{3.13}$$

We now construct z_N by perturbing the optimal vector z^* in such a way that all the unstable modes of the system are zeroed at time N. Theorem B.1 shows that such a z_N exists and it is feasible with respect to all constraints when $N > N'$ for some positive N'. After stage N, the input is set to zero, and z_N is as follows

$$z_N = (v_{N,0}, v_{N,1}, \dots, v_{N,N-1}, 0, \dots).$$

Clearly $z_N \in \bar{\mathcal{C}}_N$. We can write

$$\|z_N - z^*\|^2 \le \sum_{j=0}^{N-1} \|v_{N,j} - v_j^*\|_2^2 + \sum_{j=N}^{\infty} \|v_j^*\|_2^2$$

From (B.14) and (B.15), we have that the first term goes to zero as $N \to \infty$ and, from (3.13), we have that also the second term goes to zero as $N \to \infty$. We conclude that $\|z_N - z^*\| \to 0$ as $N \to \infty$ and hence that $z^* \in \mathrm{cl}(\bar{\mathcal{C}})$, as claimed.

Having verified that the assumptions of Theorem A.5 in Appendix A.2 are satisfied, we can now apply this theorem to deduce that the sequence of minimizers $\bar{z}_N$ of the upper bounding problem

(3.1) (alternatively, the problem obtained by replaying $\mathcal{C}$ by $\bar{\mathcal{C}}_N$ in (3.11)), approaches the solution of the optimal problem (2.7), and that the sequence of objective values converges monotonically to the optimal objective value. That is, we have

$$\lim_{N\to\infty} \bar{z}_N = z^*, \qquad \Phi_N^u \downarrow \Phi^*. \tag{3.14}$$

From (3.12) and (3.14), we have that the solutions $\hat{z}_N$ and $\bar{z}_N$ of $\mathcal{L}(N)$ and $\mathcal{U}(N)$, respectively, approach the solution z^* of the optimal problem (2.7) as $N \to \infty$. Moreover, we can apply the result of Theorem A.3 to obtain an estimate of the distance of the first component $\hat{v}_{N,0}$ from the solution $\hat{z}_N$ of $\mathcal{L}(N)$ to the corresponding optimal component v_0^*. In Theorem A.3, let the canonical problem (A.1) correspond to the "most relaxed" problem—the lower bounding problem $\mathcal{L}(N)$ in (3.6). We have from this result that

$$\frac{1}{2}\alpha\|\hat{z}_N - z^*\|^2 \le \Phi^* - \Phi_N^l \le \Phi_N^u - \Phi_N^l,$$

where we use the fact that $\Phi_N^u \ge \Phi^*$ in the second inequality, and the constant $\alpha > 0$ is discussed in Appendix A.1; this gives a useful bound, since Φ^* itself is not available. Thus we deduce that

$$\|\hat{v}_{N,0} - v_0^*\| \le \|\hat{z}_N - z^*\| \le \left[\frac{2}{\alpha}(\Phi_N^u - \Phi_N^l)\right]^{1/2}. \tag{3.15}$$

4 Implementation issues

The results of the previous section suggest an iterative approach to determining an approximation to the v_0^* component of the solution of the infinite horizon problem (2.7). In this approach, we solve a series of quadratic programs for the upper and the lower bound problems (3.1) and (3.6). If the difference between the optimal objective values for these problems does not satisfy a chosen stopping criterion, the horizon is increased; otherwise the first input v_0^l of the computed sequence of the lower bound problem (3.6) is accepted as a good approximation to v_0, and is injected into the plant.

As stopping criterion we use a relative difference between the upper and the lower bound solution:

$$\frac{\Phi_N^u - \Phi_N^l}{1 + \Phi_N^l} \le \rho, \tag{4.1}$$

where ρ is a small positive number.

At each sampling time we apply the following algorithm, starting with a positive horizon $N > 0$.

1. Solve (3.3). If the problem is infeasible, go to 5. Otherwise, let Φ_N^u be the optimal value of the objective function.

2. If the final state w_N does not belong to the output admissible set for constraints inactive at steady state [9], go to 5.

3. Solve (3.8). Let Φ_N^l be the optimal value of the objective function.

4. Check (4.1). If satisfied, go to 6.

5. Increment the horizon N and go to 1.

6. Set v_0 equal to the first solution component of (3.8).

The proposed algorithm always terminates because from (3.12) and (3.14) we have that

$$\lim_{N\to\infty} \Phi_N^u - \Phi_N^l = 0, \tag{4.2}$$

which implies that for any $\rho > 0$ there exists a N' such that for $N > N'$ the stopping criterion (4.1) is satisfied.

Theorem A.3 and (3.15) provide a measure of the distance from the input of the lower bound solution (3.6) and of the optimal solution (2.7). In (3.15) the monotonicity constant α appears in the denominator, so the bound is tighter when α is large. A numerical algorithm for computing a bound on α is available but we omit it for the sake of brevity.

5 Case study

In this section we present a numerical example of a system in which the control action does not become permanently active or inactive at steady state.

The following system is considered:

$$x_{k+1} = \begin{bmatrix} 0.5477 & 0.8208 & 0 \\ -0.8208 & 0.5067 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_k, \tag{5.1}$$

$$y_k = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x_k. \tag{5.2}$$

The controller is required to drive the system state to the origin, from the initial state $x_0 = \begin{bmatrix} 3 & 3 & 0 \end{bmatrix}^T$. The control action is bounded as follows:

$$u_k \leq 0. \tag{5.3}$$

Tuning parameters for the controller are $Q = 1$ and $R = 1$.

For this system the target values for both input and state are zero, which is also a bound for the input. Hence, the origin lies on the boundary of the feasible region. The unconstrained control law $u_k = -Kx_k$ for this system requires both positive and negative inputs. As shown in Figure 2, the corresponding constrained control action does not become permanently active or inactive. A logarithmic scale is used in the figure to emphasize this behavior.

The relative tolerance between the upper bound and the lower bound solutions in (4.1) is chosen equal to $\rho_1 = 10^{-6}$. In Figure 1, we plot the horizon length against the relative difference between the upper bound and the lower bound objective functions, obtained at time $k = 0$. This plot shows that a horizon of about 70 is needed to satisfy the chosen tolerance ρ_1. This relative tolerance guarantees a precise convergence of the entire input sequence $(u_0, u_1, \ldots)$ to the optimal value. However, even with a bigger tolerance (and therefore a shorter horizon), the injected input u_0 is still close to the optimal value. In fact, in Figure 2 the injected input is reported for the chosen tolerance ρ_1 and for the larger tolerance $\rho_2 = 10^{-3}$. These two input sequences are close to each other except for the last part of the simulation, where the magnitude of the input is small in any case. The output for the closed-loop system is reported in Figure 3, both for the tolerance ρ_1 and ρ_2. The two lines are essentially indistinguishable, showing that even a weaker convergence tolerance produced a response close to the optimal one.

6 Conclusions

In this paper the existence and the implementation of the infinite horizon controller for the case of active steady-state constraints has been discussed. This case is important because, in practical applications, controllers are often required to operate at the boundary of the feasible region. For this case only suboptimal solutions were available, based on finite horizon formulations with terminal equality constraints or infinite horizon formulations with appropriate suboptimal finite parameterization. An iterative algorithm was presented that determines the optimal solution of this problem within a user specified tolerance. Since the optimal infinite horizon solution is found, the proposed controller is simple to understand and tune, and achieves better performance than the previous solutions.

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A Quadratic Programs and Sequences of Sets: Definitions and Theorems

A.1 Convex Quadratic Programs in ℓ^2

We consider the space ℓ^2, which is the infinite-dimensional set of objects of the form:

$$z = (z_1, z_2, z_3, \ldots), \quad z_i \in \mathbb{R} \text{ for all } i = 1, 2, \ldots.$$

such that

$$\sum_{i=1}^{\infty} z_i^2 < \infty.$$

When equipped with the following inner product:

$$\langle a, b\rangle = \sum_{i=1}^{\infty} a_i b_i,$$

ℓ^2 is a separable Hilbert space [16]. We define the norm on this space in the obvious way:

$$\|z\| = \langle z, z\rangle^{1/2}.$$

Consider the following convex optimization problem over ℓ^2:

$$\min_z f(z) = \frac{1}{2}\langle z, Uz\rangle + \langle c, z\rangle, \qquad \text{subject to } z \in \mathcal{C}, \tag{A.1}$$

where

- $U : \ell^2 \to \ell^2$ is a linear, self-adjoint, strictly monotone operator; that is, there is $\alpha > 0$ such that

$$\langle z, Uz\rangle \geq \alpha\langle z, z\rangle, \qquad \text{for all } z \in \ell^2. \tag{A.2}$$

- $\mathcal{C} \subset \ell^2$ is convex, closed, and nonempty.

We define the normal cone to $\mathcal{C}$ at a point $\bar{z}$ as follows:

$$N_{\mathcal{C}}(\bar{z}) = \{v \mid \langle v, \bar{z} - z\rangle \geq 0, \;\; \text{for all } z \in \mathcal{C}\}. \tag{A.3}$$

Theorem A.1 *If z^* solves* (A.1)*, then we have*

$$-(Uz^* + c) \in N_{\mathcal{C}}(z^*), \tag{A.4}$$

and in particular, we have

$$\langle Uz^* + c, z - z^*\rangle \geq 0, \qquad \text{for all } z \in \mathcal{C}. \tag{A.5}$$

Proof. We can apply the Corollary on p. 52 of Clarke [17]. At the solution z^*, we have:

$$0 \in \partial f(z^*) + N_{\mathcal{C}}(z^*), \tag{A.6}$$

where ∂f denotes the generalized gradient of f at z^*. Since f is quadratic and U has the properties described above, we have $\partial f(z^*) = \{Uz^* + c\}$, so that (A.4) follows immediately from (A.6). The second claim follows from (A.3). □

Theorem A.2 *The problem* (A.1) *has a unique minimizer z^*.*

Proof. Given any $z_0 \in \mathcal{C}$, the level set

$$\{z \,|\, f(z) \leq f(z_0)\}$$

is closed and bounded, by the monotonicity property (A.2). Hence f attains a minimum on this set, say at z^*. Uniqueness of z^* is also a consequence of monotonicity. $\square$

Theorem A.3 *Let z^* be the solution of* (A.1)*. Then for all other $z \in \mathcal{C}$, we have that*

$$\|z - z^*\| \leq \left[2\frac{f(z) - f(z^*)}{\alpha}\right]^{1/2}. \tag{A.7}$$

Proof. Using (A.5), we have $\langle c, z - z^* \rangle \geq -\langle Uz^*, z - z^* \rangle$. Hence, we have

$$\begin{aligned} f(z) - f(z^*) &= \frac{1}{2}\langle z, Uz \rangle - \frac{1}{2}\langle z^*, Uz^* \rangle + \langle c, z - z^* \rangle \\ &\geq \frac{1}{2}\langle z, Uz \rangle - \frac{1}{2}\langle z^*, Uz^* \rangle - \langle Uz^*, z - z^* \rangle = \frac{1}{2}\langle z - z^*, U(z - z^*) \rangle \\ &\geq \frac{1}{2}\alpha \|z - z^*\|^2, \end{aligned}$$

where the last inequality follows from (A.2). $\square$

A.2 Increasing and Decreasing Sequences of Sets

We now consider monotonic sequences of subsets of ℓ^2, their limits, and the behavior of the sequence of points obtained by minimizing the function $f(z)$ defined in (A.1) over each of the sets in these sequences.

Consider first a decreasing sequence of sets $\{\hat{\mathcal{C}}_J\}_{J=1,2,\dots}$ such that

$$\text{each } \hat{\mathcal{C}}_J \subset \ell^2 \text{ is closed, convex, and nonempty;} \tag{A.8}$$

$$\hat{\mathcal{C}}_1 \supset \hat{\mathcal{C}}_2 \supset \cdots. \tag{A.9}$$

This sequence has a limit $\hat{\mathcal{C}}$ defined by

$$\hat{\mathcal{C}} = \bigcap_{J=1,2,\dots} \hat{\mathcal{C}}_J; \tag{A.10}$$

see [16, p. 66]. It is clear that $\hat{\mathcal{C}}$ too is closed, convex, and nonempty. $\hat{\mathcal{C}}$ is simply the set of points that belong to every set $\hat{\mathcal{C}}_J$ in the sequence. We also have the following characterization.

Lemma A.1

$$\hat{\mathcal{C}} = \{z \,|\, z = \lim_{J\to\infty} z_J, \textit{ for any convergent sequence } \{z_J\} \textit{ with } z_J \in \hat{\mathcal{C}}_J \textit{ for all } J\}. \tag{A.11}$$

Proof. Assume first that $z \in \hat{\mathcal{C}}$. The trivial sequence defined by $z_J \equiv z$ suffices to show that z belongs to the set on the right-hand side of (A.11).

Now assume that $z \notin \hat{\mathcal{C}}$. We show that there can exist no sequence $\{z_J\}$ with $z_J \in \hat{\mathcal{C}}_J$, all J with the property that $\|z_J - z\| \to 0$.

Since $z \notin \hat{\mathcal{C}}$, we have that $z \notin \hat{\mathcal{C}}_K$ for some K and indeed, by closedness of $\hat{\mathcal{C}}_K$, we have $\text{dist}(z, \hat{\mathcal{C}}_K) > 0$. Therefore, by the monotonicity property of the sequence $\{\hat{\mathcal{C}}_J\}$, we have that $z \notin \hat{\mathcal{C}}_J$ for all $J \geq K$, and in fact that

$$\text{dist}(z, \hat{\mathcal{C}}_J) \geq \text{dist}(z, \hat{\mathcal{C}}_K) > 0, \quad \text{for all } J \geq K.$$

It follows that for *any* sequence $\{z_J\}$ with the property $z_J \in \hat{\mathcal{C}}_J$, we have that

$$\|z - z_J\| \geq \text{dist}(z, \hat{\mathcal{C}}_J) \geq \text{dist}(z, \hat{\mathcal{C}}_K) > 0, \quad \text{for all } J \geq K,$$

so we cannot have $\|z - z_J\| \to 0$. $\square$

We consider the sequence of problems $\hat{P}(J)$ defined as follows

$$\hat{P}(J): \quad \min_z f(z) \text{ subject to } z \in \hat{\mathcal{C}}_J, \tag{A.12}$$

where $f(z)$ is defined as in (A.1). By applying Theorem A.1, we can identify points $\hat{z}_J$, $J = 1, 2, \ldots$ such that $\hat{z}_J$ is the unique solution of $\hat{P}(J)$ for each J. Similarly, we define $\hat{z}$ to be the unique minimizer of f over the limiting set $\hat{\mathcal{C}}$. By the decreasing property (A.9), and the definition (A.10), we have that

$$f(\hat{z}_1) \leq f(\hat{z}_2) \leq \cdots \leq f(\hat{z}). \tag{A.13}$$

In fact, we have the following result.

Theorem A.4 *For $\hat{z}_J$, $J = 1, 2, \ldots$ and $\hat{z}$ defined in the previous paragraph, we have*

$$\lim_{J \to \infty} \hat{z}_J = \hat{z}. \tag{A.14}$$

Proof. The sequence of real numbers

$$\{f(\hat{z}_J)\}_{J=1,2,\ldots} \tag{A.15}$$

is increasing and bounded above. Using the following argument, we can show that this sequence $\{\hat{z}_J\}$ is Cauchy. Given any indices J_1 and J_2 with $J_1 < J_2$, we have that $\hat{z}_{J_2}$ is feasible in $P(J_1)$. Hence, by applying Theorem A.3 to $P(J_1)$ with $z^* = \hat{z}_{J_1}$ and $z = \hat{z}_{J_2}$, we have that

$$\|\hat{z}_{J_1} - \hat{z}_{J_2}\| \leq \left[2 \frac{f(\hat{z}_{J_1}) - f(\hat{z}_{J_2})}{\alpha} \right]^{1/2}.$$

Hence, for any $\epsilon > 0$, there exists $J_{(\epsilon)}$ such that

$$\|\hat{z}_{J_1} - \hat{z}_{J_2}\| \leq \epsilon, \quad \text{for all } J_1,\ J_2 \text{ with } J_{(\epsilon)} \leq J_1 < J_2.$$

Because of this Cauchy property and the fact that ℓ^2 is a Hilbert space, the sequence $\{\hat{z}_J\}$ converges to a limit in ℓ^2, say z^*. In fact, because of the characterization (A.11), we have $z^* \in \hat{\mathcal{C}}$. Because $f(\hat{z}_J) \uparrow f(z^*)$, we have from (A.13) that $f(z^*) \leq f(\hat{z})$. But since $z^* \in \hat{\mathcal{C}}$ and $\hat{z}$ is the unique minimizer of f over $\hat{\mathcal{C}}$, we must have $\hat{z} = z^*$, completing the proof. $\square$

We next consider an increasing sequence of sets. Let $\{\bar{\mathcal{C}}_J\}_{J=1,2,\ldots}$ be a sequence of sets such that

$$\text{each } \bar{\mathcal{C}}_J \subset \ell^2 \text{ is closed, convex, and nonempty;} \tag{A.16}$$

$$\bar{\mathcal{C}}_1 \subset \bar{\mathcal{C}}_2 \subset \cdots . \tag{A.17}$$

This sequence has a limit $\bar{\mathcal{C}}$ defined by

$$\bar{\mathcal{C}} = \bigcup_{J=1,2,\ldots} \bar{\mathcal{C}}_J; \tag{A.18}$$

see [16, p. 66]. The set $\bar{\mathcal{C}}$ is convex and nonempty but *not* necessarily closed. As an example, consider the sets defined by

$$\bar{\mathcal{C}}_J = \{w = (w_1, w_2, w_3, \ldots) \,|\, w \in \ell^2, \ w_i = 0 \text{ for all } i \geq J\},$$

which yield an increasing sequence whose limit is

$$\bar{\mathcal{C}} = \{w = (w_1, w_2, w_3, \ldots) \,|\, w \in \ell^2, \ w_i = 0 \text{ for all } i \text{ sufficiently large}\}.$$

Although the sets $\bar{\mathcal{C}}_J$ are closed for all J, the limit $\bar{\mathcal{C}}$ is open. The point $w = (1, 1/2, 1/4, 1/8, \ldots)$ lies in the closure of $\bar{\mathcal{C}}$ though not in $\bar{\mathcal{C}}$ itself.

Since the limit $\bar{\mathcal{C}}$ may be an open set, the function $f(z)$ may not attain its minimizer on this set. We can still however show convergence of the sequence of minimizers of f over $\bar{\mathcal{C}}_J$ to a point $\bar{z}$ that minimizes f over some closed set containing $\bar{\mathcal{C}}$, which we denote by $\mathcal{C}^*$.

Similarly to (A.12), we consider the sequence of problems $\bar{P}(J)$ defined as follows

$$\bar{P}(J): \quad \min_z f(z) \text{ subject to } z \in \bar{\mathcal{C}}_J, \tag{A.19}$$

where $f(z)$ is defined as in (A.1). By applying Theorem A.1, we can identify points $\bar{z}_J$, $J = 1, 2, \ldots$ such that $\bar{z}_J$ is the unique solution of $\bar{P}(J)$ for each J. We also define a point $\bar{z}$ and a set $\mathcal{C}^*$ as follows:

$$\bar{z} = \arg\min\nolimits_{z \in \mathcal{C}^*} f(z), \quad \text{where } \mathcal{C}^* \text{ is a set satisfying} \tag{A.20a}$$

$$\mathcal{C}^* \text{ is closed}, \ \ \bar{\mathcal{C}} \subset \mathcal{C}^*, \ \text{ and } \bar{z} \in \mathrm{cl}(\bar{\mathcal{C}}). \tag{A.20b}$$

(Note that since $\mathcal{C}^*$ is closed and $\bar{\mathcal{C}} \subset \mathcal{C}^*$, we certainly have $\mathrm{cl}(\bar{\mathcal{C}}) \subset \mathcal{C}^*$.) Clearly, we have that

$$f(\bar{z}_1) \geq f(\bar{z}_2) \geq \cdots \geq f(\bar{z}). \tag{A.21}$$

In fact, we have the following result.

Theorem A.5 *For $\bar{z}_J$, $J = 1, 2, \ldots$ and $\bar{z}$ defined in the previous paragraph, we have*

$$\lim_{J\to\infty} \bar{z}_J = \bar{z}. \tag{A.22}$$

Proof. We can show that the sequence $\{\bar{z}_J\}$ is Cauchy by using as similar argument as in the proof of Theorem A.4. Hence the sequence converges, say to a point z^*. Moreover, since $\bar{z}_J \in \bar{\mathcal{C}}$ for all J, we have that $z^* \in \mathrm{cl}(\bar{\mathcal{C}})$.

Because of (A.21), we have that $f(z^*) \geq f(\bar{z})$ Suppose for the moment that this inequality is strict. Since $\bar{z} \in \mathrm{cl}(\bar{\mathcal{C}})$, there is a sequence $\{y_K\}$ with $y_K \in \bar{\mathcal{C}}$ for all K, such that $y_K \to \bar{z}$. By the definition (A.18), we can choose indices K and J_K sufficiently large that the following properties hold:

$$f(y_K) < f(z^*), \tag{A.23}$$

$$y_K \in \bar{\mathcal{C}}_J, \text{ for all } J \geq J_K. \tag{A.24}$$

In particular, we have that

$$f(y_K) < f(z^*) \leq f(\bar{z}_{J_K}),$$

which is a contradiction since $\bar{z}_{J_K}$ is the minimizer of f over $\bar{\mathcal{C}}_{J_K}$. Therefore, we must have $f(z^*) = f(\bar{z})$.

Since $\bar{z}$ is the unique minimizer of f over the set $\mathcal{C}^*$, it certainly is the minimizer of f over $\mathrm{cl}(\bar{\mathcal{C}})$. Since $z^* \in \mathrm{cl}(\bar{\mathcal{C}})$ and $f(z^*) = f(\bar{z})$, we have $z^* = \bar{z}$, completing the proof. $\square$

B Existence of the upper bounding problem

We show that the existence of an infinite feasible sequence for the optimal problem (2.7) implies that the upper bounding optimization problem (3.3) is feasible for finite N. In the upper bounding optimization problem (3.3), the use of the linear control law $v_k = -\bar{K}w_k$ requires the controller to zero in finite time the unstable modes that are not controllable in the null space of the active steady-state constraints. We show here the stronger result that the existence of an infinite feasible sequence for the optimal problem (2.7) implies we can zero all unstable modes in finite time.

Without loss of generality, we make two simplifying assumptions. First, consider a Schur decomposition of the system matrix partitioned into stable and unstable parts. We consider only the unstable part, i.e. we consider a purely unstable system, and show we can zero the entire state in finite time.

Second, we assume that the initial state w_0 is sufficiently close to the origin that all constraint boundaries that do not intersect the origin (i.e. constraints that are not active at steady state) appear arbitrarily far away and can be neglected. This assumption is valid since, for any sequence $\{w_k, v_k\}$ feasible for (2.7), we have that $\lim_{k\to\infty} w_k = 0$, $\lim_{k\to\infty} v_k = 0$ and, therefore, the system state can be brought arbitrarily close to the origin in finite time.

B.1 Preliminary definitions

Definition B.1 (General problem) *We consider the following problem:*

$$w_0 \text{ given}, \qquad w_{k+1} = Aw_k + Bv_k \quad k = 0, 1, \ldots, \tag{B.1a}$$

$$\bar{D}v_k \leq 0 \quad k = 0, 1, \ldots, \tag{B.1b}$$

in which $A \in \mathbb{R}^{n \times n}$ has all eigenvalues outside the open unit circle, i.e. $|\lambda_i(A)| \geq 1$, $i = 1, \ldots, n$, $B \in \mathbb{R}^{n \times m}$ and $\bar{D} \in \mathbb{R}^{m_a \times m}$, with $m_a \leq m$, and $\operatorname{rank}(\bar{D}) = m_a$. We assume that a sequence $\{w_k, v_k\}_{k=0}^{\infty}$ exists such that (B.1a) *and* (B.1b) *are satisfied and*

$$\sum_{k=0}^{\infty} w_k^T Q w_k + v_k^T R v_k < \infty \tag{B.2}$$

for Q, R symmetric positive definite matrices.

Without loss of generality, we assume that $\bar{D} = \begin{bmatrix} I_{m_a} & 0 \end{bmatrix}$. The transformation of the input u and, consequently, of the matrix B that leads to this form is

$$v \leftarrow Tv = \begin{bmatrix} \bar{D} \\ \bar{D}^c \end{bmatrix} v, \qquad B \leftarrow BT^{-1}, \qquad R \leftarrow T^{-T}RT^{-1}$$

in which $\bar{D}^c \in \mathbb{R}^{(m-m_a) \times m}$ is such that $T \in \mathbb{R}^{m \times m}$ is invertible.

Lemma B.1 *Given a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$, (A, B) is controllable iff (A^{-1}, B) is controllable.*

Proof. If (A, B) is controllable we have that:

$$\begin{aligned} n &= \operatorname{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \operatorname{rank} \left(A^{n-1} \begin{bmatrix} A^{-n+1}B & A^{-n+2}B & \cdots & B \end{bmatrix} \right) \\ &= \operatorname{rank} \begin{bmatrix} B & A^{-1}B & \cdots & A^{-(n-1)}B \end{bmatrix}, \end{aligned}$$

which implies that (A^{-1}, B) is controllable. Necessity is proven in an analogous way. □

Definition B.2 *We define the controllability matrix of order k for (A^{-1}, B) as*

$$\mathbb{C}_k = \begin{bmatrix} B & A^{-1}B & A^{-2}B & \cdots & A^{-(k-1)}B \end{bmatrix}. \tag{B.3}$$

We also define the infinite dimensional controllability matrix for (A^{-1}, B) as

$$\mathbb{C}_\infty = \lim_{k \to \infty} \mathbb{C}_k. \tag{B.4}$$

The matrix $\mathbb{C}_\infty$ has bounded elements since A has unstable eigenvalues.

Definition B.3 *We define the matrix* $\mathbb{D}_k \in \mathbb{R}^{m_a k \times mk}$ *as*

$$\mathbb{D}_k = \begin{bmatrix} \bar{D} & 0 & \cdots & 0 \\ 0 & \bar{D} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{D} \end{bmatrix},$$

and the infinite dimensional matrix $\mathbb{D}_\infty$ *as:*

$$\mathbb{D}_\infty = \lim_{k\to\infty} \mathbb{D}_k.$$

B.2 Main proof

Let $\pi_\infty = (v_0, v_1, \cdots) \in \ell^2$ be an input vector that satisfies (B.1a), (B.1b) and (B.2). We can write

$$w_k - A^k w_0 = A^{k-1}Bv_0 + A^{k-2}Bv_1 + \cdots + Bv_{k-1},$$

or alternatively

$$A^{-(k-1)}w_k - Aw_0 = Bv_0 + A^{-1}Bv_1 + \cdots + A^{-(k-1)}Bv_{k-1}.$$

As $k \to \infty$ we have that $w_k \to 0$ and, therefore, we have

$$\mathbb{C}_\infty \pi_\infty = -Aw_0, \tag{B.5a}$$

$$\mathbb{D}_\infty \pi_\infty \leq 0. \tag{B.5b}$$

Theorem B.1 *Given an infinite vector* $\pi_\infty \in \ell^2$ *that satisfies* (B.5), *there exists a finite vector* $\bar{\pi}_N$ *for some* N *such that*

$$\mathbb{C}_N \bar{\pi}_N = -Aw_0, \tag{B.6a}$$

$$\mathbb{D}_N \bar{\pi}_N \leq 0. \tag{B.6b}$$

Proof. Some elements of the vector π_∞ may be zero. We can remove these elements from π_∞ and define a new vector $\tilde{\pi}_\infty$. Consequently, we remove the corresponding columns from $\mathbb{C}_\infty$ and $\mathbb{D}_\infty$ obtaining new matrices $\tilde{\mathbb{C}}_\infty$ and $\tilde{\mathbb{D}}_\infty$. If $\tilde{\pi}_\infty$ has a finite number of elements, the proof is complete because we can construct $\bar{\pi}_N$ from π_∞ by choosing N such that $v_j = 0,\ j > N$. If all elements of π_∞ are zero, the proof is also complete because $w_0 = 0$. We assume, therefore, that $\tilde{\pi}_\infty$ has an infinite number of elements. We can rewrite (B.5) as

$$\tilde{\mathbb{C}}_\infty \tilde{\pi}_\infty = -Aw_0, \tag{B.7a}$$

$$\tilde{\mathbb{D}}_\infty \tilde{\pi}_\infty < 0. \tag{B.7b}$$

in which the strict inequality comes from the structure of $\mathbb{D}_\infty$ and from the fact that the zero elements of π_∞ have been removed in $\tilde{\pi}_\infty$.

Let $l \leq n$ be the rank of $\tilde{\mathbb{C}}_\infty$. We have that there exists a M such that $\text{rank}(\mathbb{C}_J) = l$ for any $J \geq M$. From (B.7a) we also have that $\text{rank}([\tilde{\mathbb{C}}_\infty | Aw_0]) = l$. Given a $N > M$, we partition the vector $\tilde{\pi}_\infty$ and the matrices $\tilde{\mathbb{C}}_\infty$ and $\tilde{\mathbb{D}}_\infty$ as follows

$$\tilde{\pi}_\infty = (\tilde{\pi}_N | \tilde{\pi}_{N|\infty}), \quad \tilde{\mathbb{C}}_\infty = [\tilde{\mathbb{C}}_N | \tilde{\mathbb{C}}_{N|\infty}], \quad \tilde{\mathbb{D}}_\infty = [\tilde{\mathbb{D}}_N | \tilde{\mathbb{D}}_{N|\infty}].$$

From (B.7a) we have

$$\tilde{\mathbb{C}}_N \tilde{\pi}_N + \tilde{\mathbb{C}}_{N|\infty} \tilde{\pi}_{N|\infty} = -Aw_0 \tag{B.8}$$

We wish to construct a finite dimensional vector π_N as $\pi_N = \tilde{\pi}_N + \rho_N$ in which

$$\rho_N = (v_0, v_1, \ldots, v_{M-1}, 0, \ldots, 0)$$

such that

$$\tilde{\mathbb{C}}_N \pi_N = -Aw_0, \tag{B.9a}$$

$$\tilde{\mathbb{D}}_N \pi_N < 0. \tag{B.9b}$$

Using (B.7a), we have that

$$\tilde{\mathbb{C}}_N (\tilde{\pi}_N + \rho_N) = -Aw_0 = \tilde{\mathbb{C}}_N \tilde{\pi}_N + \tilde{\mathbb{C}}_{N|\infty} \tilde{\pi}_{N|\infty},$$

from which

$$\tilde{\mathbb{C}}_N \rho_N = \tilde{\mathbb{C}}_{N|\infty} \tilde{\pi}_{N|\infty}. \tag{B.10}$$

Since the last $N - M$ terms of ρ_N are zero, we can rewrite (B.10) as

$$\tilde{\mathbb{C}}_M \rho_M = \tilde{\mathbb{C}}_{N|\infty} \tilde{\pi}_{N|\infty} \tag{B.11}$$

Using (B.9b) instead, we obtain

$$\tilde{\mathbb{D}}_N \rho_N < -\tilde{\mathbb{D}}_N \tilde{\pi}_N \tag{B.12}$$

Since $N > M$, we can take a sub-matrix of (B.12) and use the particular structure of $\tilde{\mathbb{D}}_N$ to obtain:

$$\tilde{\mathbb{D}}_M \rho_M < -\tilde{\mathbb{D}}_M \tilde{\pi}_M. \tag{B.13}$$

Since $N > M$, we have that $\tilde{\mathbb{C}}_{N|\infty} \tilde{\pi}_{N|\infty} \in \text{range}(\tilde{\mathbb{C}}_N) = \text{range}(\tilde{\mathbb{C}}_M)$ and, therefore, (B.11) admits solution. Let $\tilde{\mathbb{C}}_M^+$ be a left inverse of $\tilde{\mathbb{C}}_M$ (*i.e.* $\tilde{\mathbb{C}}_M^+ \tilde{\mathbb{C}}_M = I$). One solution of (B.11) is:

$$\rho_M = \tilde{\mathbb{C}}_M^+ \tilde{\mathbb{C}}_{N|\infty} \tilde{\pi}_{N|\infty}. \tag{B.14}$$

Since $\tilde{\mathbb{C}}_{N|\infty} \tilde{\pi}_{N|\infty} \to 0$ as $N \to \infty$, we have that

$$\lim_{N \to \infty} \|\rho_M\|_2 = 0 \tag{B.15}$$

Hence, since $-\tilde{\mathbb{D}}_M \tilde{\pi}_M > 0$ in (B.13), there exists N' such that (B.13) holds for all $N \geq N'$. Choosing any $N \geq N'$, we have found a vector π_N that satisfies (B.9a) and (B.9b). We can obtain the vector $\bar{\pi}_N$ that satisfies (B.6a) and (B.6b) by reinserting the zero elements that have been removed from π_∞ to obtain $\tilde{\pi}_\infty$. $\square$

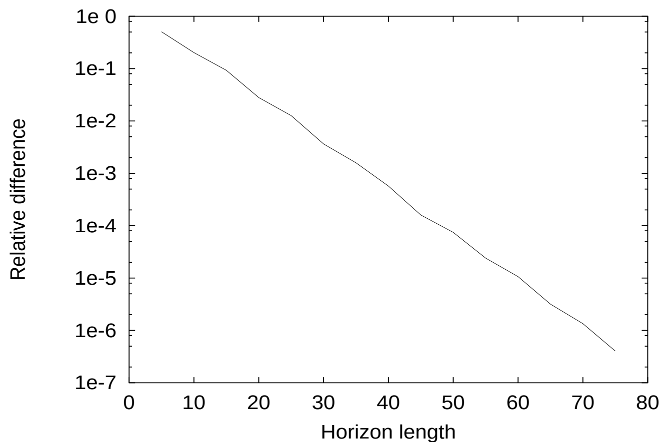

Figure 1: Horizon length *vs* relative difference of objective functions

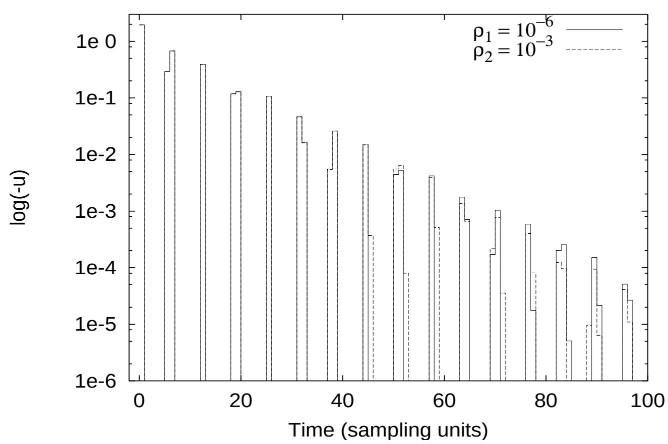

Figure 2: Closed-loop input for Example # 1

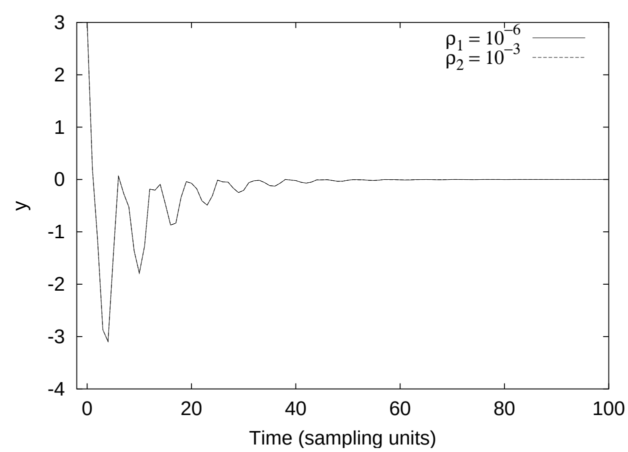

Figure 3: Closed-loop output for Example # 1